# Logical Error Tolerance of Tree-Encoding

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NOTE1: The numerical results shown in Table 1 are not entirely correct because of the binomial coefficient calculation bug in the code.

NOTE2: Section 3 has been modified from v2020. Previous version was not entirely correct if we use the strategy that we trust the indirect measurement result even when a photon is not lost. This changes the tree repeater error analysis.

It has been shown that encoding a logical quantum bit with a tree state can tolerate a photon loss of up to 50% [1]. Interestingly, as also qualitatively mentioned in [1], the tree state also has some logical error tolerance. This benefits from the large redundancy in the tree structure. Note that the tree state can NOT totally correct the logical error, but only suppress the error sensitivity due to the large overhead in the state. To explain this argument, let's consider an example. Assume an identical error probability $\varepsilon$ for all qubits in the tree, the upper bound of the error probability of the tree can be given by $e_{\text{upper}} = 1-(1-\varepsilon)^N$, where $N$ is the number of qubits in the tree. This upper bound means that as long as there is a qubit in the tree has an error, the whole tree fails. This would significantly limit the size of the tree that can be useful, such that limit the loss error correction capability. Fortunately, this upper bound is actually a far overestimation. The error probability of the tree can be as small as only several times as large as the single qubit error, as we will see later. Therefore, as long as we can make the single-qubit error probability small enough, the error of the tree can be neglected, and we are relatively safe to consider large trees. This is crucial since in the photonic cluster state case, dealing with loss is the most important thing, and this requires large trees.

More specifically, depending on what we use the tree state for, the error tolerance analysis can be different. For example, if we use tree-encoding for RGS-based quantum repeaters, the only single-qubit measurements we need to perform are the $X$- and $Z$-measurements. We know the cluster state is the eigenstate of its stabilizer, so these two kinds of measurements can have a significant error tolerance, as analyzed in [2, 3, 4]. On the other hand, if we use tree-encoding for measurement-based quantum computation (MBQC), where arbitrary single-qubit measurement is needed, the error tolerance can be not as good as the $X$- and $Z$-measurement cases, but can be still way below the upper bound. Later we will also see that for the tree-repeater, the error tolerance is actually exactly the same as in MBQC, since the level-1 photon that is used for spin-photon CZ gate plays the same role as the level-1 qubit in MBQC which is applied by an arbitrary measurement.

This error tolerance mainly come from two aspects. First, since there are many branches attached to a non-bottom qubit, in the case where a success of any of the branches heralds the success of an indirect measurement on the parent node, we can use the majority vote strategy to suppress the effect of error. This applies to the indirect $Z$-measurement scenario, and is due to the fact that the tree state is the only eigenstate of the stabilizer with eigenvalue 1. Second, in many cases, the result of an indirect measurement only depends on the parity of a set of measurements. So only if there are odd number of errors will the indirect measurement gets wrong. We will illustrate these two aspects more explicitly later.

Previous works have provided many insightful analysis. In [1], a qualitative argument is given for an arbitrary single-qubit measurement. In [2, 3], an analytical calculation of the "effective error probability" for $X$- and $Z$-measurements in the presence of loss is provided. In [4], a numerical simulation of the effective error probability is performed by considering logical and loss errors at the same time. We will repeat the mathematical details of the analytical calculations, mainly following [3], in this note, and apply the same technique to analyze the arbitrary single-qubit measurement and tree-repeater cases.

# 1 Elementary strategies

## 1.1 Stabilizer formalism

The error tolerance of the tree state benefits from the fact that a cluster state is the only eigenstate of its stabilizer, with the eigenvalue 1. To verify this, let's consider a simple example, as shown in Fig. 1. This

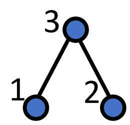


Figure 1: A cluster state is the only eigenstate of its stabilizer, with the eigenvalue 1.

small tree can be mathematically expressed as

$$|\psi\rangle_{\text{tree}} = |0++\rangle_{312} + |1--\rangle_{312} = |+\rangle_3 (|00\rangle + |11\rangle)_{12} + |-\rangle_3 (|01\rangle + |10\rangle)_{12}. \tag{1}$$

The stabilizer of this tree state is $K = X_3 Z_1 Z_2$, so

$$\begin{aligned} K|\psi\rangle_{\text{tree}} &= X_3 Z_1 Z_2 \left[|+\rangle_3 (|00\rangle + |11\rangle)_{12} + |-\rangle_3 (|01\rangle + |10\rangle)_{12}\right] \\ &= |+\rangle_3 (|00\rangle + |11\rangle)_{12} + |-\rangle_3 (|01\rangle + |10\rangle)_{12} \\ &= |\psi\rangle_{\text{tree}}. \end{aligned} \tag{2}$$

In other words, we can retrieve a single $X$- or $Z$-measurement outcome by considering the parity of all other measurements in the same stabilizer. Therefore, the indirect $Z$-measurement introduced in [1] is possible. Actually, the stabilizer code is one of the important error correction codes in quantum information science.

## 1.2 Majority vote

Let's now discuss how the majority vote strategy works. Consider a depth-2 tree with branching parameters $\{3, 2\}$, as shown in Fig. 2, and we want to perform an indirect $Z$-measurement on the root qubit 10. As

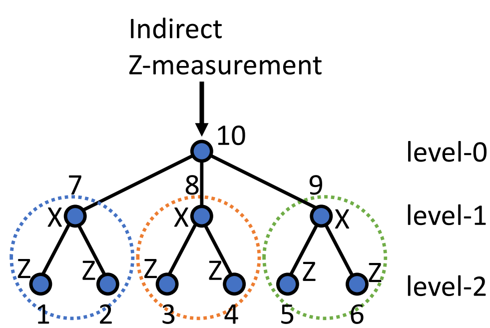


Figure 2: An indirect $Z$-measurement succeeds if any of the branches can be successfully measured.

we know from [1, 5], this indirect $Z$-measurement succeeds if **any** of the $X$-$Z$-$Z$ measurement sets succeeds for the three subtrees. Note that if one of the subtrees is successfully measured, there is no need to detach the other two. This is because the $X$-measurement has already done this [6].

Our goal is to get the correct $Z$-measurement outcome of qubit 10. If qubit 10 itself is not lost, we can directly do this, with some error probability. If, unfortunately, it is lost, we can choose one of the three branches, where all the level-1 and level-2 qubits in this branch are not lost. As long as such a measurement set on a branch succeeds, the other two branches are also detached from qubit 10, and are left in the same state as the measured branch. This key fact tells us that if the other two branches can be also performed by the same measurement set, they should give us the same indirect $Z$-measurement result as the measured branch did. Therefore, as long as we can, we should perform the same measurement set on all the branches, in case there are some errors. Suppose we are able to do this on all the three branches, and two of them give us an indirect $Z$-measurement outcome 0, while the third branch gives a result 1, according to the majority vote principle, we will say the indirect $Z$-measurement outcome is 0.

To see this mathematically, we can write down the state of this tree (not normalized)

$$|\psi\rangle_{\text{tree}} = |0\rangle_{10}\left(|0++\rangle_{712} + |1--\rangle_{712}\right)^{\otimes 3} + |1\rangle_{10}\left(|0++\rangle_{712} - |1--\rangle_{712}\right)^{\otimes 3}. \tag{3}$$

Then suppose we first measure the blue-circled (Fig. 2) branch, and get the results $\{X_7, Z_1, Z_2\} = \{+, 0, 0\}$. This heralds that qubit 10 is now in state $|0\rangle_{10}$. The tree state is projected to

$$|\psi\rangle_{\text{tree}} \to |0\rangle_{10}\left(|{+}00\rangle_{712}\right)^{\otimes 3} = |0\rangle_{10} \otimes |{+}00\rangle_{712} \otimes |{+}00\rangle_{834} \otimes |{+}00\rangle_{956}. \tag{4}$$

If we do the measurements on the other two branches, the outcome should be $\{X_8, Z_3, Z_4\} = \{X_9, Z_5, Z_6\} = \{+, 0, 0\}$, ideally. However, due to some unexpected errors, the projected state may become

$$|\psi\rangle_{\text{tree}} \to |1\rangle_{10} \otimes |{+}00\rangle_{712} \otimes |{+}01\rangle_{834} \otimes |{+}01\rangle_{956}. \tag{5}$$

In this case, if we still speculate the state of qubit 10 based on the measurement outcomes $\{X_7, Z_1, Z_2\} = \{+, 0, 0\}$, it would be a wrong answer. But if we use the majority vote strategy, it is clear that we can still retrieve the correct $Z$-measurement outcome of qubit 10.

### 1.3 Parity

Another mechanism for error correction using tree states benefits from the fact that we decide the indirect measurement outcome based on parity but not the explicit results. To see this, we still consider the depth-3 tree as an example. As we have already discussed, if a branch gives results $\{X_7, Z_1, Z_2\} = \{+, 0, 0\}$, we say qubit 10 is in state $|0\rangle_{10}$. Furthermore, if we get a set of results $\{X_7, Z_1, Z_2\} = \{+, 1, 1\}$ in the absence of error, the tree state is projected to

$$|\psi\rangle_{\text{tree}} \to |0\rangle_{10}\left(|{+}11\rangle_{712}\right)^{\otimes 3}, \tag{6}$$

so we also say qubit 10 is in state $|0\rangle_{10}$. But if we get this measurement result due to some errors, the real projected state could be

$$|\psi\rangle_{\text{tree}} \to |0\rangle_{10} \otimes |{+}11\rangle_{712} \otimes |{+}00\rangle_{834} \otimes |{+}00\rangle_{956}. \tag{7}$$

In this case, although we have some errors in the measurement results, we can still get the correct answer, because $|{+}00\rangle$ and $|{+}11\rangle$ have the same parity.

## 2 Error tolerance in tree-encoded RGS

Making use of the two aspects above, we can then analyze the effective error probability of a single $X$- or $Z$- measurement in the RGS quantum repeater protocol [2, 3]. Here we do the calculations in the presence of loss, and use the loss analysis already derived in [1, 5]. **For simplicity, we assume an identical error probability for all the qubits in the tree when directly measured, denoted by** $\epsilon$. We know from Eq. 38 in [5] that the success probability of an indirect $Z$-measurement on a level-$k$ qubit is given by

$$r_k = 1 - (1 - s_k)^{b_k}\text{, with } s_k = (1-\mu)(1 - \mu + \mu r_{k+2})^{b_{k+1}}, \tag{8}$$

where $s_k$ is the success probability that one of the branches succeeds, and $\mu$ is the single-photon loss rate.

In the quantum repeater protocol using RGS, we know that to do entanglement swapping, if a Bell-measurement on a second-leaf-photon pair succeeds, we need to perform two $X$-measurements on the corresponding first-leaf photons to extend the entanglement distance. If a Bell-measurement fails or not needed any more, we need to perform two $Z$-measurements on the corresponding first-leaf photons to detach them. When we encode the first-leaf photons with tree structures, these two kinds of measurement on the logical qubits need separate treatments, although there are many similarities. This is because when we need to perform a $Z$-measurement on a logical qubit, it succeeds if **all** the $Z$-measurements (either direct or indirect) on the first-leaf photons succeed. While when we need to perform a $X$-measurement on a logical qubit, it succeeds if **any** of the $X$-measurements on the first-leaf photons succeeds. This is illustrated in Fig. 3.

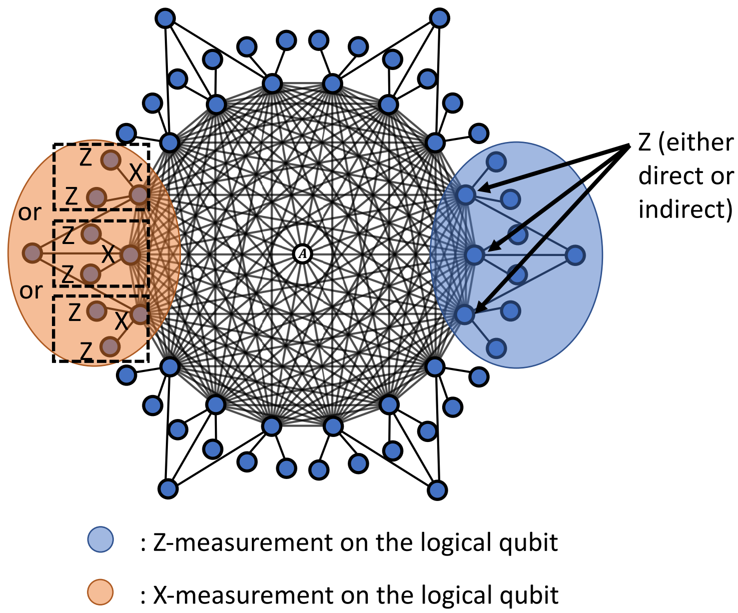


Figure 3: A $Z$-measurement on a logical qubit succeeds if all the 1st-leaf qubits are successfully measured (either direct or indirect) under $Z$-basis. A $X$-measurement on a logical qubit succeeds if any of the $X$-$Z$-$Z$ measurement sets is successfully performed (the $Z$-measurement can be indirect if the encoding tree is deeper).

It can be easily understood that if we need to perform a $Z$-measurement on a logical qubit, which means we want to detach this logical qubit, we must detach all the branches by performing $Z$-measurements on all the level-1 qubits. But it might be a bit confusing that a $X$-measurement on a logical qubit only requires one of the branches successfully measured, especially when we consider the other half of RGS in the same measurement node. This can be verified with a simple example. Suppose we've come to the last step of creating an entangled pair between Alice and Bob. In the logical-qubit representation, it is easy: a double-$X$-measurement will detach the two logical qubits and connect Alice and Bob, as illustrated in Fig. 4(a). What about we think in the tree-encoding representation [Fig. 4(b)]?

Suppose we encode each logical qubit with a $\{2, 2\}$ tree, then explicitly, if we choose to perform the double-$X$-measurement on the upper-left and lower-right branches, we can verify that the other two branches will be detached and Alice and Bob are also finally connected, as shown in Fig. 4(c). Note that we use the fact that performing a double-$X$-measurement on two connected qubits has the effect of connecting each qubit in the neighborhood of the first qubit to all the qubits in the neighborhood of the second one. Also, if there are two qubits on different side who are already connected, they will be disconnected after the double-$X$-measurement. This is because if we apply the CZ gate on two qubits twice, they will be left unchanged.

## 2.1 $Z$-measurement on a logical qubit

For a $Z$-measurement on the logical qubit, we require all the $b_0$ level-1 qubits are successfully measured under $Z$-basis (possibly with error). These $Z$-measurements can be either direct or indirect. Therefore, the

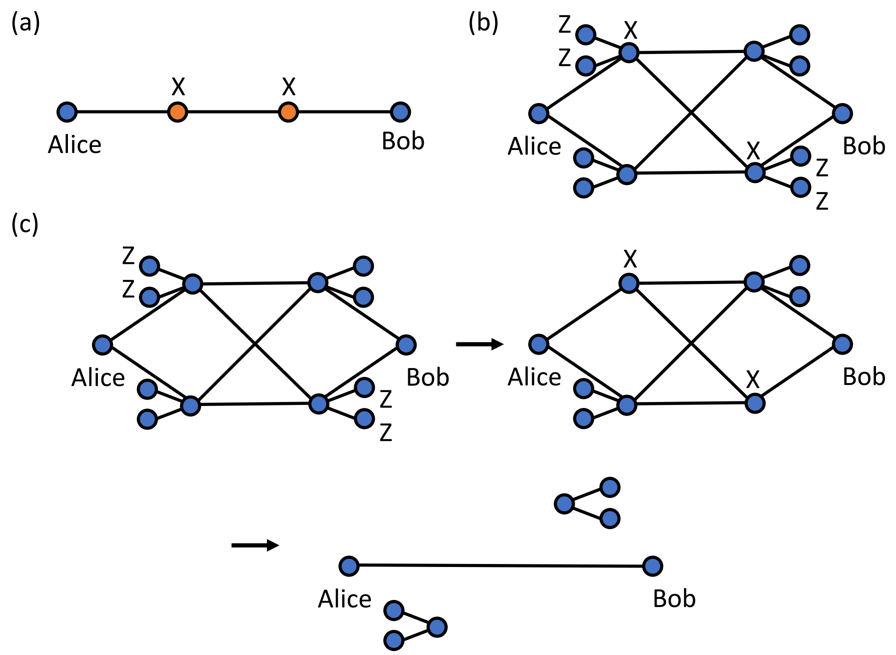


Figure 4: A double-$X$-measurement on the two logical qubits connects Alice and Bob. (a) The logical qubit representation. (b) The tree-encoding representation when using a $\{2,2\}$ tree. (c) The process of the double-$X$-measurement.

success probability of this $Z$-measurement on the logical qubit can be given by

$$\begin{aligned} &P(Z\text{-meas. on the logical qubit succeeds}) \\ =&P(Z\text{-meas. on all level-1 photons succeed}) \\ =&[P(\text{a direct or indirect } Z\text{-meas. succeeds})]^{b_0} \\ =&\boxed{(1-\mu+\mu r_1)^{b_0}}. \end{aligned} \tag{9}$$

We use the strategy that if a level-1 qubit is not lost, we try both the direct (must succeed since photon not lost) and indirect (may succeed) $Z$-measurement and if the indirect measurement succeeds, we choose to believe its outcome instead of taking the direct outcome. This is because we believe with the error correction, the indirect measurement outcome has a smaller error probability compared with the directly measured outcome. While if it is lost, we use (the only hope) an indirect $Z$-measurement. In other words, we may encounter four circumstances

$$\begin{aligned} &\text{Photon lost}\begin{cases}\text{Indirect measurement succeeds, prob} = \mu r_1 & \leftarrow \text{case1}\\ \text{Indirect measurement fails, prob} = \mu(1-r_1), & \leftarrow \text{case2}\end{cases}\\ &\text{Photon survives}\begin{cases}\text{Indirect measurement succeeds, prob} = (1-\mu)r_1 & \leftarrow \text{case3}\\ \text{Indirect measurement fails, prob} = (1-\mu)(1-r_1). & \leftarrow \text{case4}\end{cases} \end{aligned} \tag{10}$$

Now, the question is, given such a $Z$-measurement succeeds, what is the averaged error probability $\bar{e}_Z$? Since some of the $Z$-measurements on level-1 photons are direct, and some are indirect, let's first calculate the corresponding conditional probabilities,

$$\begin{aligned} P(I_1|Z_1) &= P(\text{prob. of an indirect } Z\text{-meas. on a level-1 qubit given a } Z\text{-meas. succeeds}) \\ &= P(\text{success prob. of an indirect } Z\text{-meas.})/P(\text{success prob. of a } Z\text{-meas.}) \\ &= r_1/(1-\mu+\mu r_1). \end{aligned} \tag{11}$$

Then, we have

$$\boxed{\bar{e}_Z = \sum_{n=0}^{b_0} C_{b_0}^n P(I_1|Z_1)^{b_0-n}[1 - P(I_1|Z_1)]^n e_n}, \tag{12}$$

where $n$ is the number of direct measurements, $e_n$ is the error probability in the case there are $n$ direct measurements and $(b_0 - n)$ indirect ones performed among the total $b_0$ measurements. Then $e_n$ is given by

$$e_n = \sum_{i=0}^{n} \left[ C_n^i \epsilon^i (1-\epsilon)^{n-i} \sum_{\substack{j=0,\\ i+j=1[2]}}^{b_0-n} C_{b_0-n}^j e_{I_1}^j (1 - e_{I_1})^{b_0-n-j} \right], \tag{13}$$

where $i$ is the number of errors among $n$ direct measurements, $j$ is the number of errors among $(b_0-n)$ indirect measurements. $\epsilon$ is the error probability of a single direct measurement, and $e_{I_1}$ is the error probability of an indirect measurement on a level-1 qubit. In the second sum, we have the condition $i + j = 1[2]$ because only an incorrect parity gives an error.

Before we proceed to evaluate $e_{I_1}$, let's make ourselves comfortable on using the conditional probability $P(I_1|Z_1)$ to calculate $\bar{e}_Z$. Looking back to Eq. 10, if we calculate $\bar{e}_Z$ directly, we will have

$$\bar{e}_Z = \frac{P_{\text{wrong}}}{P_{\text{success}}}, \tag{14}$$

where $P_{\text{success}}$ is given by Eq. 9, and $P_{\text{wrong}}$ is the probability that we get a wrong result (a successful measurement is not guaranteed), which can be calculated by

$$P_{\text{wrong}} = \sum_{n=0}^{b_0} C_{b_0}^n \left[(1-\mu)(1-r_1)\right]^n r_1^{b_0-n} e_n, \tag{15}$$

where $n$ is again the number of using direct measurement results, and $e_n$ is given by the same equation, i.e. Eq. 13. Note that in Eq. 10, the corresponding error probabilities for each case is

$$\begin{cases} \text{case 1:} & e_{I_1}, \\ \text{case 2:} & \text{fails, no outcome,} \\ \text{case 3:} & e_{I_1}, \\ \text{case 4:} & \epsilon. \end{cases} \tag{16}$$

Therefore the term $(1-\mu)(1-r_1)$ in Eq. 15 comes from case 4, and the term $r_1^{b_0-n}$ comes from the combination of case 1 and case 3. Their corresponding error probabilities $\epsilon$ and $e_{I_1}$ are mapped to the expression of $e_n$. Now, is Eq. 14 same as Eq. 12? The answer is yes, which can be easily verified. From Eq. 14, we have

$$\begin{aligned} \bar{e}_Z &= \frac{P_{\text{wrong}}}{P_{\text{success}}} \\ &= \frac{\sum_{n=0}^{b_0} C_{b_0}^n \left[(1-\mu)(1-r_1)\right]^n r_1^{b_0-n} e_n}{(1-\mu+\mu r_1)^{b_0}} \\ &= \sum_{n=0}^{b_0} C_{b_0}^n e_n \frac{(1-\mu)^n (1-r_1)^n r_1^{b_0-n}}{(1-\mu+\mu r_1)^{b_0}}, \end{aligned} \tag{17}$$

and from Eq. 12, we have

$$\begin{aligned} \bar{e}_Z &= \sum_{n=0}^{b_0} C_{b_0}^n e_n P(I_1|Z_1)^{b_0-n}[1 - P(I_1|Z_1)]^n \\ &= \sum_{n=0}^{b_0} C_{b_0}^n e_n \left(\frac{r_1}{1-\mu+\mu r_1}\right)^{b_0-n} \left(1 - \frac{r_1}{1-\mu+\mu r_1}\right)^n \\ &= \sum_{n=0}^{b_0} C_{b_0}^n e_n \frac{(1-\mu+\mu r_1 - r_1)^n r_1^{b_0-n}}{(1-\mu+\mu r_1)^{b_0}}. \end{aligned} \tag{18}$$

This gives the same result.

Now we are ready to proceed with the calculation of $e_{I_1}$. To evaluate $e_{I_1}$, let's consider the more general case, $e_{I_k}$, the error probability of an indirect $Z$-measurement on a level-$k$ qubit, given it succeeds. We know that this indirect measurement succeeds if any of the $X$-$Z$-$Z$ measurement sets ($b_k$ in total) succeeds, and depending on how many such measurement sets we can perform, the power of majority vote strategy is different. Suppose there are $m_k$ such measurement sets succeed, with a probability $p_k(m_k)$, we have

$$\begin{aligned} r_k e_{I_k} &= P(\text{success \& wrong}) = \sum_{m_k=1}^{b_k} p_k(m_k) e_{I_k|m_k}, \\ \Rightarrow \quad e_{I_k} &= \frac{1}{r_k} \sum_{m_k=1}^{b_k} p_k(m_k) e_{I_k|m_k}, \end{aligned} \tag{19}$$

where $e_{I_k|m_k}$ is the error probability of the indirect measurement given there are $m_k$ branches measured, and

$$p_k(m_k) = C_{b_k}^{m_k} s_k^{m_k} (1-s_k)^{b_k-m_k}. \tag{20}$$

Moving on, given there are $m_k$ branches measured, it gives a wrong answer of the indirect measurement outcome only if there are more than a half giving wrong answers, so

$$\begin{aligned} e_{I_k|m_k} &= P(N_{\text{errors}} > m_k/2) \\ &= \begin{cases} \sum_{j=(m_k+1)/2}^{m_k} C_{m_k}^j e_{I_k|1}^j (1-e_{I_k|1})^{m_k-j} & m_k \text{ odd}, \\ \sum_{j=m_k/2}^{m_k-1} C_{m_k-1}^j e_{I_k|1}^j (1-e_{I_k|1})^{m_k-1-j} & m_k \text{ even}, \end{cases} \end{aligned} \tag{21}$$

where $e_{I_k|1}$ is the averaged error probability of a single indirect measurement set. When $m_k$ is even, we randomly drop a result and make $(m_k-1)$ odd.

Now the last question is how we calculate $e_{I_k|1}$. We know that the $X$-measurement on the level-$(k+1)$ photon has an error probability $\epsilon$, and the $b_{k+1}$ $Z$-measurements on level-$(k+2)$ can be either direct or indirect. Therefore, we have

$$e_{I_k|1} = \sum_{n_k=0}^{b_{k+1}} C_{b_{k+1}}^{n_k} P(I_{k+2}|Z_{k+2})^{b_{k+1}-n_k} [1-P(I_{k+2}|Z_{k+2})]^{n_k} e_{n_k}, \tag{22}$$

where $n_k$ is the number of direct measurements on level-$(k+2)$, and $e_{n_k}$ is the error probability given there are $n_k$ direct measurements on level-$(k+2)$. Note that this $n_k$ has a similar definition as the $n$ in Eq. 12, but $e_{n_k}$ is slightly different than the $e_n$ before. Here, $e_{n_k}$ also needs to consider the error in the level-$(k+1)$ $X$-measurement, but $e_n$ doesn't. So we cannot say $e_n = e_{n_{-1}}$, although $n_{-1} = n$ is true. So let's finish the last step,

$$e_{n_k} = \sum_{i=0}^{n_k+1} \left[ C_{n_k+1}^i \epsilon^i (1-\epsilon)^{n_k+1-i} \sum_{\substack{j=0, \\ i+j=1[2]}}^{b_{k+1}-n_k} C_{b_{k+1}-n_k}^j e_{I_{k+2}}^j (1-e_{I_{k+2}})^{b_{k+1}-n_k-j} \right], \tag{23}$$

where $i$ is the number of errors in the total $(n_k+1)$ direct measurements ($n_k$ direct $Z$ and 1 direct $X$), $j$ is the number of error s in the total $(b_{k+1}-n_k)$ indirect measurements in level-$(k+2)$, and $e_{I_{k+2}}$ is the error probability of an indirect measurement on level-$(k+2)$ given it succeeds.

Therefore, in a recursive way, we are able to calculate the error probability of a $Z$-measurement on a logical encoded 1st-leaf qubit of an RGS, given it succeeds.

## 2.2 $X$-measurement on a logical qubit

For a $X$-measurement on a logical qubit, we only require one of the branches can be successfully measured in the $X$-$Z$-$Z$ pattern. But in the spirit of majority vote, we should measure as many branches as we can.

The success probability of such a $X$-measurement can be given by

$$\begin{aligned}
&P(X\text{-meas. on the logical qubit succeeds})\\
=&P(\text{any of the branches is successfully measured in the } X\text{-}Z\text{-}Z \text{ pattern})\\
=&1-[P(\text{the meas. on a branch fails})]^{b_0}\\
=&1-[1-P(\text{the meas. on a branch succeeds})]^{b_0}\\
=&1-[1-P(\text{the } X\text{-meas. on level-1 qubit succeeds})P(\text{the } Z\text{-meas. on all corresponding level-2 qubits succeed})]^{b_0}\\
=&\boxed{1-[1-(1-\mu)(1-\mu+\mu r_1)^{b_1}]^{b_0}}.
\end{aligned} \tag{24}$$

Actually, this is just the success probability of an indirect $Z$-measurement on the root qubit of the tree, although the root qubit it self doesn't even exist in the tree-encoded RGS.

Then, given such a $X$-measurement succeeds, the error probability can be given in the similar way as Eq. 19, i.e.,

$$\boxed{\bar{e}_X = e_{I_0} = \frac{1}{r_0}\sum_{m_0=1}^{b_0} p_0(m_0) e_{I_0|m_0}, \text{where } r_0 = P(X\text{-meas. on the logical qubit succeeds})}. \tag{25}$$

## 3 Error tolerance in tree-repeaters

In the one-way quantum repeater protocol [4], we can model the error into $\epsilon_r$ in the re-encoding step. This error can come from the generation process, and the spin-photon CZ gate in the re-encoding. It is numerically shown (in part IV B. "Logical errors") that if we only consider the error originating from the tree state, $\epsilon_r$ will depend on the size of the encoding, and $\epsilon_r/\epsilon \approx 3-5$ given an identical error probability $\epsilon$ for all the qubits in the tree. In this case, $\epsilon_r$ is just equivalent to the effective error probability defined in the last section. It was also noticed in [4] that this relation doesn't depend on the tree structures but just the size. This can be explained in the sense that the error mainly come from the level-1 photon which participating the spin-photon CZ gate, and the rest can be significantly suppressed through the majority vote and parity strategies. This is actually very similar to the MBQC case, where the error mainly come from the level-1 qubit which is applied by an arbitrary measurement. As we will see later, all the calculations in this section can be directly applied to the MBQC case, i.e., they are actually the same thing mathematically.

First, let's calculate the success probability of a re-encoding operation, considering a deterministic spin-photon CZ gate. Then the success probability is just the probability that we can decode the transmitted quantum bit. This succeeds if the following three conditions can be satisfied:

- there are at least one level-1 photon that is not lost;
- the corresponding level-2 photons of the chosen level-1 photon can be successfully $Z$-measured, either directly or indirectly;
- all other level-1 photons can be successfully $Z$-measured, either directly or indirectly.

One may have noticed that these three conditions are exactly those required in the MBQC scenario! Therefore, again, all the calculations in this section can be directly applied to the MBQC case.

Therefore, the success probability of a re-encoding step, or technically the first part of re-encoding which is just decoding, can be given by (Eq. 5 in [4] & Eq. 37 in [5])

$$\boxed{P(\text{success of decoding}) = \left[(1-\mu+\mu R_1)^{b_0} - (\mu R_1)^{b_0}\right](1-\mu+\mu R_2)^{b_1}}, \tag{26}$$

where $\mu, R_i$ all have the same definitions as before. The second term $(1-\mu+\mu R_2)^{b_1}$ clearly means the success probability of $Z$ measurements in level-2. The first term comes from

$$\begin{aligned}
&(1-\mu+\mu R_1)^{b_0}-(\mu R_1)^{b_0}\\
=&(1-\mu)^{b_0}+C_{b_0}^1\mu R_1(1-\mu)^{b_0-1}+C_{b_0}^2(\mu R_1)^2(1-\mu)^{b_0-2}+\cdots+C_{b_0}^{b_0-1}(\mu R_1)^{b_0-1}(1-\mu)\\
=&(1-\mu)^{b_0}+C_{b_0}^1\mu R_1(1-\mu)^{b_0-1}+\cdots+C_{b_0}^{b_0-1}(\mu R_1)^{b_0-1}(1-\mu)+(\mu R_1)^{b_0}-(\mu R_1)^{b_0}\\
=&(1-\mu+\mu R_1)^{b_0}-(\mu R_1)^{b_0}.
\end{aligned} \tag{27}$$

By the way, based on the conditions to be satisfied above, we also need to choose one surviving photon for decoding. How does this fact show itself in the calculations above? Actually for each term in the second line of Eq. 27, it should be $(1-\mu)^{b_0}C_{b_0}^1\frac{1}{b_0}$, and $C_{b_0}^2(\mu R_1)^2(1-\mu)^{b_0-2}C_{b_0-1}^1\frac{1}{b_0-1}$, and so on, where $C_{b_0}^1$ and $C_{b_0-1}^1$ mean we have $b_0$ and $b_0-1$ choices for this decoding level-1 photon, while $\frac{1}{b_0}$ and $\frac{1}{b_0-1}$ represent the probability of each surviving photon to be selected.

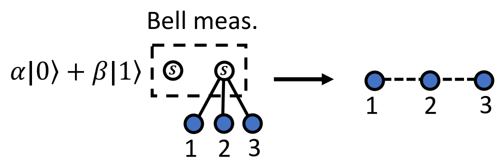


Figure 5: A $b_0=3$ tree is used to encode a qubit $\alpha|0\rangle+\beta|1\rangle$.

So, the more important question is that what is the averaged effective error probability $\bar{e}_{\text{decoding}}$, given such a decoding step succeeds? Before answering this question, one may concern that if the parity strategy still works in this tree-repeater case, since as we have seen in [7] that all level-1 photons are entangled in a slightly different way than normal. Let's now prove the parity strategy still works. Considering the simplest case, as shown in Fig. 5, where we use a depth-1 tree with $b_0=3$, and the qubit that needs to be sent is $\alpha|0\rangle+\beta|1\rangle$. Then after the Bell measurement in the encoding step, the three level-1 photons are in state

$$|\psi\rangle=\alpha|+++\rangle_{123}+\beta|---\rangle_{123}. \tag{28}$$

Suppose all the three qubits are not lost, and we choose qubit 1 for the spin-photon CZ gate. This means that we should perform $Z$-measurements on qubits 2 and 3 to detach them. We have

$$\begin{aligned}
|\psi\rangle&=\alpha|+++\rangle_{123}+\beta|---\rangle_{123}\\
&=\alpha|+\rangle_1(|00\rangle+|01\rangle+|10\rangle+|11\rangle)_{23}+\beta|-\rangle_1(|00\rangle-|01\rangle-|10\rangle+|11\rangle)_{23}\\
&=(\alpha|+\rangle+\beta|-\rangle)_1(|00\rangle+|11\rangle)_{23}+(\alpha|+\rangle-\beta|-\rangle)_1(|01\rangle+|10\rangle)_{23}.
\end{aligned} \tag{29}$$

We can see that an even parity of the $Z$-measurement outcomes on 2 and 3 means qubit 1 is projected to state $(\alpha|+\rangle+\beta|-\rangle)$, while an odd parity projects qubit 1 to state $(\alpha|+\rangle-\beta|-\rangle)$. So it is true that only a wrong parity will result in an error.

Now, to calculate the effective error probability, let's use the direct way instead of conditional probability, i.e., we evaluate the decoding error probability without the commitment of a successful decoding. Therefore

$$\bar{e}_{\text{decoding}}=\frac{P(\text{we get a wrong result})}{P(\text{success of decoding})}, \tag{30}$$

where $P$(success of decoding) is already given by Eq. 26. What is left is to calculate $P$(we get a wrong result). We have

$$\begin{aligned}
&P(\text{we get a wrong result})\\
=&\boxed{\sum_{l=0}^{b_0-1}\sum_{n=0}^{b_0-l}\sum_{m=0}^{b_1}C_{b_0}^{l}C_{b_0-l}^{n}C_{b_1}^{m}(\mu R_1)^l[(1-\mu)(1-R_1)]^n[(1-\mu)R_1]^{b_0-l-n}[(1-\mu)(1-R_2)]^m R_2^{b_1-m}\bar{e}_{m,n}},
\end{aligned} \tag{31}$$

where $l$ is the number of photons situated in the first level of the tree that are physically lost but the indirect $\sigma_Z$ measurements on them can succeed, $n$ is the number of photons situated in the first level of the tree that survive but the indirect $\sigma_Z$ measurements on them will fail, and $(b_0-l-n)$ is the number of photons situated in the first level of the tree that survive and the indirect $\sigma_Z$ measurements on them can succeed. Note that we adopt a strategy that if both direct and indirect $\sigma_Z$ measurements can be performed on a photon, we choose the indirect result for decoding since the error correction capability of the tree encoding should give a smaller error probability. Therefore, one photon out of the $(b_0-l)$ surviving first-level photons will be used for entangling with the matter qubit and the $\sigma_X$ measurement for decoding, $n$ first-level photons will provide the direct $\sigma_Z$ measurement outcomes, and $(b_0-n-1)$ first-level photons will provide the indirect $\sigma_Z$ measurement outcomes. Similarly, for the $b_1$ second-level photons that are entangled with the $\sigma_X$-measured first-level photon, $m$ is the number of photons that survive but the indirect $\sigma_Z$ measurements on them fail, and $(b_1-m)$ is the number of photons on which the indirect $\sigma_Z$ measurements succeed (they can be physically lost or not lost). $\bar{e}_{n,m}$ is the average error probability of the decoded logical qubit in the case in which $n$ measurement outcomes under the $\sigma_Z$ basis from the first level and $m$ measurement outcomes under the $\sigma_Z$ basis in the second level of the tree are from the direct measurements, while all other $\sigma_Z$ measurement outcomes are obtained from the indirect measurements.

To calculate $e_{m,n}$, we can think of $1-e_{m,n}$, the probability that we get a correct result in the case of $m,n$. This success requires the decoding qubit is correct, the $(b_0-1)$ $Z$ measurements on the rest level-1 photons give the correct parity, and the $b_1$ $Z$ measurements on level-2 photons of the chosen branch give the correct parity. Actually, this being said, how do we know we need both parities to be correct or we just need the overall parity of $(b_0-1+b_1)$ $Z$ measurements to be correct? The answer is we do need both parities to be correct. Let's prove this before we proceed. Consider a $\{3,3\}$ tree as shown in Fig. 6, the state of the whole tree is

$$\begin{aligned}
&|\psi\rangle = |0\rangle_S \left(|0\rangle_1 |{+}{+}{+}\rangle_{234} + |1\rangle_1 |{-}{-}{-}\rangle_{234}\right)^{\otimes 3} + |1\rangle_S \left(|0\rangle_1 |{+}{+}{+}\rangle_{234} - |1\rangle_1 |{-}{-}{-}\rangle_{234}\right)^{\otimes 3} \\
\text{encoding} \Rightarrow \quad & \alpha(|0\rangle_1 |{+}{+}{+}\rangle_{234} + |1\rangle_1 |{-}{-}{-}\rangle_{234})^{\otimes 3} + \beta(|0\rangle_1 |{+}{+}{+}\rangle_{234} - |1\rangle_1 |{-}{-}{-}\rangle_{234})^{\otimes 3} \\
&\equiv \alpha |\psi_+\rangle |\psi_+\rangle |\psi_+\rangle + \beta |\psi_-\rangle |\psi_-\rangle |\psi_-\rangle , \quad \text{with} \quad |\psi_\pm\rangle = |0\rangle |{+}{+}{+}\rangle \pm |1\rangle |{-}{-}{-}\rangle \\
&= \alpha(|0\rangle_1 |{+}{+}{+}\rangle_{234} + |1\rangle_1 |{-}{-}{-}\rangle_{234}) |\psi_+\rangle |\psi_+\rangle + \beta(|0\rangle_1 |{+}{+}{+}\rangle_{234} - |1\rangle_1 |{-}{-}{-}\rangle_{234}) |\psi_-\rangle |\psi_-\rangle \\
&= \alpha(|+\rangle_1 |\text{even 1's}\rangle_{234} + |-\rangle_1 |\text{odd 1's}\rangle_{234}) |\psi_+\rangle |\psi_+\rangle + \beta(|-\rangle_1 |\text{even 1's}\rangle_{234} + |+\rangle_1 |\text{odd 1's}\rangle_{234}) |\psi_-\rangle |\psi_-\rangle \\
&= |\text{even 1's}\rangle_{234} \left(\alpha |+\rangle_1 |\psi_+\rangle |\psi_+\rangle + \beta |-\rangle_1 |\psi_-\rangle |\psi_-\rangle\right) + |\text{odd 1's}\rangle_{234} \left(\alpha |-\rangle_1 |\psi_+\rangle |\psi_+\rangle + \beta |+\rangle_1 |\psi_-\rangle |\psi_-\rangle\right),
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
|\text{even 1's}\rangle &= |{+}{+}{+}\rangle + |{-}{-}{-}\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle , \\
|\text{odd 1's}\rangle &= |{+}{+}{+}\rangle - |{-}{-}{-}\rangle = |001\rangle + |010\rangle + |100\rangle + |111\rangle .
\end{aligned} \tag{33}$$

Now for decoding, we measure photons 2-6 under $Z$ basis. First, after measuring photons 2, 3, 4, depending on the measured parity, we have

$$\text{measure 2, 3, 4} \Rightarrow \begin{cases} \text{even parity} & \alpha |+\rangle_1 |\psi_+\rangle |\psi_+\rangle + \beta |-\rangle_1 |\psi_-\rangle |\psi_-\rangle \\ \text{odd parity} & \alpha |-\rangle_1 |\psi_+\rangle |\psi_+\rangle + \beta |+\rangle_1 |\psi_-\rangle |\psi_-\rangle . \end{cases} \tag{34}$$

Next, we measure photons 5, 6. Actually in $|\psi_+\rangle |\psi_+\rangle$ and $|\psi_-\rangle |\psi_-\rangle$, only photons 5 and 6 matter, we just need to remember each $|0\rangle_{5\text{ or }6}$ ($|1\rangle_{5\text{ or }6}$) should be followed by $|{+}{+}{+}\rangle$ ($|{-}{-}{-}\rangle$) as their children nodes. Thus we can just ignore level-2 photons of these two branches to simplify our expression, and replace $|\psi_+\rangle$

($|\psi_-\rangle$) with $|+\rangle$ ($|-\rangle$) respectively,

$$\begin{aligned}
\text{simplify} \Rightarrow & \begin{cases} \text{even parity} & \alpha\,|+\rangle_1|+\rangle_5|+\rangle_6 + \beta\,|-\rangle_1|-\rangle_5|-\rangle_6 \\ \text{odd parity} & \alpha\,|-\rangle_1|+\rangle_5|+\rangle_6 + \beta\,|+\rangle_1|-\rangle_5|-\rangle_6 \end{cases} \\
= & \begin{cases} \text{even parity} & \alpha\,|+\rangle_1(|00\rangle+|01\rangle+|10\rangle+|11\rangle)_{56} + \beta\,|-\rangle_1(|00\rangle-|01\rangle-|10\rangle+|11\rangle)_{56} \\ \text{odd parity} & \alpha\,|-\rangle_1(|00\rangle+|01\rangle+|10\rangle+|11\rangle)_{56} + \beta\,|+\rangle_1(|00\rangle-|01\rangle-|10\rangle+|11\rangle)_{56} \end{cases} \\
= & \begin{cases} \text{even parity} & (\alpha\,|+\rangle+\beta\,|-\rangle)_1\overbrace{(|00\rangle+|11\rangle)_{56}}^{\text{even}} + (\alpha\,|+\rangle-\beta\,|-\rangle)_1\overbrace{(|01\rangle+|10\rangle)_{56}}^{\text{odd}} \\ \text{odd parity} & (\alpha\,|-\rangle+\beta\,|+\rangle)_1\overbrace{(|00\rangle+|11\rangle)_{56}}^{\text{even}} + (\alpha\,|-\rangle-\beta\,|+\rangle)_1\overbrace{(|01\rangle+|10\rangle)_{56}}^{\text{odd}} \end{cases} \\
\text{measure 5, 6} \Rightarrow & \begin{cases} \text{even parity in 234} \begin{cases} \text{even parity in 56} & (\alpha\,|+\rangle+\beta\,|-\rangle)_1 \\ \text{odd parity in 56} & (\alpha\,|+\rangle-\beta\,|-\rangle)_1 \end{cases} \\ \text{odd parity in 234} \begin{cases} \text{even parity in 56} & (\alpha\,|-\rangle+\beta\,|+\rangle)_1 \\ \text{odd parity in 56} & (\alpha\,|-\rangle-\beta\,|+\rangle)_1. \end{cases} \end{cases}
\end{aligned} \tag{35}$$

We can now see all the four resulting states for photon 1 are different, which means that we need both parities to retrieve the correct logic qubit. QED.

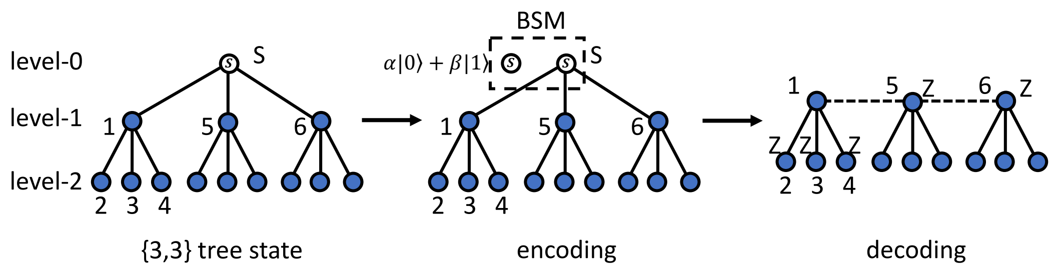


Figure 6: A $\{b_0 = 3, b_1 = 3\}$ tree is used to encode a qubit $\alpha\,|0\rangle + \beta\,|1\rangle$.

Now let's come back to the evaluation of $1 - e_{m,n}$. We have

$$\begin{aligned}
1 - e_{m,n} = \overbrace{(1-\epsilon)}^{\text{correct decoding photon}} & \left\{ 1 - \sum_{i=0}^{n} \left[ \overbrace{C_n^i \epsilon^i (1-\epsilon)^{n-i}}^{i \text{ errors in } n \text{ direct results}} \sum_{\substack{j=0,\\ i+j=1[2]}}^{b_0-1-n} \overbrace{C_{b_0-1-n}^j e_{I_1}^j (1-e_{I_1})^{b_0-1-n-j}}^{j \text{ errors in } (b_0-1-n) \text{ indirect results}} \right] \right\} \\
\times & \left\{ 1 - \sum_{i=0}^{m} \left[ \underbrace{C_m^i \epsilon^i (1-\epsilon)^{m-i}}_{i \text{ errors in } m \text{ direct results}} \sum_{\substack{j=0,\\ i+j=1[2]}}^{b_1-m} \underbrace{C_{b_1-m}^j e_{I_2}^j (1-e_{I_2})^{b_1-m-j}}_{j \text{ errors in } (b_1-m) \text{ indirect results}} \right] \right\},
\end{aligned} \tag{36}$$

where $e_{I_1}$ and $e_{I_2}$ is the error probability that a single indirect measurement in level-1 and level-2 gives an error, respectively, which is calculated in Eq.19

So far, we've had everything we need to calculate the error probability, and

$$\boxed{\bar{e}_{\text{decoding}} = \frac{P(\text{we get a wrong result})}{P(\text{success of decoding})}}. \tag{37}$$

Note that if we directly use conditional probability, we should NOT do it as

$$\text{INCORRECT} \Rightarrow \bar{e}_{\text{decoding}} = \sum_{n=0}^{b_0-1} \sum_{m=0}^{b_1} C_{b_0}^1 C_{b_0-1}^n C_{b_1}^m [1 - P(I_1|Z_1)]^n P(I_1|Z_1)^{b_0-1-n} [1 - P(I_2|Z_2)]^m P(I_2|Z_2)^{b_1-m} e_{m,n}. \tag{38}$$

This is INCORRECT because the expression for $P(I_k|Z_k)$ no longer works, since the success probability is now different, i.e., not just requiring we can perform a $Z$ measurement on single photons directly or indirectly.

## 4 Error tolerance in MBQC with tree-encoding

As has been mentioned in the last section, all the analysis in the tree-repeater case apply to the MBQC case with tree-encoding, which are Eq. 26, 30-36.

## 5 Numerical results

We can then write a code to do all the complicated calculations above recursively (see Appendix). To convince ourselves that the whole set of analysis makes sense, and to make sure we don't make trivial mistakes in the code, let's verify an example in Fig. S3 in [4]. In Fig. S3, they use numerical simulations to show that the effective error probability (denoted $\epsilon_r$ in their paper) is just a few times larger than the single-photon loss (denoted $\epsilon$), i.e., $\epsilon_r/\epsilon = 3 \sim 5$. With our code and the same tree ($\{4, 14, 4\}$), by setting $\epsilon = 0.0001$, we get $\epsilon_r = 0.000335942$, which basically agrees with the conclusion in [4]. Furthermore, by varying the total distance $L$ and corresponding optimal number of repeater stations, we always find that $\epsilon_r/\epsilon \approx 3 \sim 5$, which makes ourselves confident with the analysis.

The more interesting question we care about is if our assumption that an error smaller than $10^{-5}$ is negligible works in our paper [8]. To answer this question, we can consider the optimal trees we use in Fig. 3(b) in [8]. We choose the trees that are closest to the $\varepsilon_{\text{eff}} = 0.001$ line, and set an identical single-photon error probability $\epsilon = 10^{-5}$, the effective error probabilities for different applications are shown below. We can see that the largest ratio $(\epsilon_r/\epsilon)_{\text{max}} = 5.99621$ (marked red in 1, which is for a $Z$-measurement in tree-encoded RGS). Since we set $\epsilon = 10^{-5}$, the worst fidelity of a single-qubit measurement is still as high as $1 - 5.99621 \times 10^{-5} = 99.994\%$, which is remarkable.

Table 1: Effective error probability for some trees for different applications ($\epsilon = 10^{-5}$).

| Tree | $\{1,1,1\}$ | $\{1,1,1\}$ | $\{1,1,1\}$ | $\{2,1,1\}$ | $\{2,1,1\}$ |
|---|---|---|---|---|---|
| Single-photon loss $\mu$ | 0.0001 | 0.000178 | 0.000316 | 0.000562 | 0.001 |
| Effective loss $\mu$ | 0.000527254 | 0.00060507 | 0.000743479 | 0.000856286 | 0.00086038 |
| $Z$-meas. in RGS | 1.99978e-05 | 1.99962e-05 | 1.99935e-05 | 3.99763e-05 | 3.99589e-05 |
| $X$-meas. in RGS | 1.99998e-05 | 1.99998e-05 | 1.99998e-05 | 1.99998e-05 | 1.99998e-05 |
| Tree QR & MBQC | 1.99999e-05 | 1.99999e-05 | 1.99999e-05 | 3.00109e-05 | 3.00196e-05 |
| Tree | $\{2,1,1\}$ | $\{2,1,1\}$ | $\{2,1,1\}$ | $\{2,2,2\}$ | $\{2,2,2\}$ |
| Single-photon loss $\mu$ | 0.00178 | 0.00316 | 0.00562 | 0.01 | 0.0178 |
| Effective loss $\mu$ | 0.000873315 | 0.00091415 | 0.0010429 | 0.000647272 | 0.000951053 |
| $Z$-meas. in RGS | 3.99279e-05 | 3.98729e-05 | 3.97758e-05 | 5.99621e-05 | 5.98891e-05 |
| $X$-meas. in RGS | 1.99998e-05 | 1.99998e-05 | 1.99998e-05 | 2.99994e-05 | 2.99994e-05 |
| Tree QR & MBQC | 3.00351e-05 | 3.00623e-05 | 3.01103e-05 | 4.0395e-05 | 4.06963e-05 |
| Tree | $\{3,3,2\}$ | $\{3,5,3\}$ | $\{4,10,4\}$ | $\{6,10,9,1\}$ | $\{8,31,28,2\}$ |
| Single-photon loss $\mu$ | 0.0316 | 0.00562 | 0.1 | 0.178 | 0.316 |
| Effective loss $\mu$ | 0.00143567 | 0.000926442 | 0.000756504 | 0.000927333 | 0.00120089 |
| $Z$-meas. in RGS | 2.25611e-05 | 7.55423e-06 | 2.87413e-06 | 5.57221e-06 | 2.2454e-09 |
| $X$-meas. in RGS | 3.68942e-06 | 2.86473e-06 | 7.97165e-07 | 1.25704e-07 | 1.48565e-07 |
| Tree QR & MBQC | 5.97631e-05 | 3.67114e-05 | 4.15217e-05 | 5.17074e-05 | 5.47058e-05 |

We also note that intuitively, the effective error probability for tree-repeater and MBQC applications will be larger than those for an $X$- or $Z$-measurement in a tree-encoded RGS. But this NOT always true, as can be told from our results. This is because for the $Z$-measurement on a logical qubit in RGS case, we need to consider the errors of all level-1 qubits, while in tree-repeater or MBQC case, we also need to do so, except that we separate the chosen qubit for special use. Therefore, it is not surprising that these two cases have similar performance. On the other hand, for the $X$-measurement in RGS, the effective error of such a measurement depends on the majority vote of all level-1 results. Compared with the parity dependence in the other two cases, it is not surprising either that they should have similar performance.

**In conclusion, all the three cases have similar performance in terms of error tolerance, where the effective error probability is just of the same order of magnitude (or better) as the single-photon error probability. This conclusion shows that the tree structure can also have pretty strong logical error correction capability, besides the well-known loss error correction capability**.

## Appendices

Code for numerical calculations of effective error probability: https://github.com/Y-Zhan/PerformanceAnalysisOfQR

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[6] Also note that not just the $Z$-measurement can detach a qubit from the graph, other measurements can also do this, but in different ways. For example, the double-$X$ measurement can detach the corresponding two adjacent qubits and fully connect the rest; a $X$-measurement together with some $Z$-measurement can have the effect of indirect $Z$-measurement, which disconnect all qubits that are originally connected with the indirectly-measured qubit; a $Y$-measurement detaches a qubit and fully connects those who are originally connected to the measured one, up to some local rotations, which are equivalent to the local complementation operation.

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